

Universal Aspects of Two-Dimensional Yang-Mills Theory at Large N

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ABSTRACT

We show that the large N partition functions and Wilson loop observables of two-dimensional Yang-Mills theories admit a universal functional form irrespective of the gauge group. We demonstrate that $U(N)$ QCD₂ undergoes a large N , third-order phase transition on the projective plane at an area-coupling product of $\pi^2/2$. We use this as a lemma to provide a direct transcription of the partition functions and phase portraits of Yang-Mills theory from the $U(N)$ on \mathbf{RP}^2 at large N to the other classical Lie groups on S^2 . We compute the expectation value of the Wilson loops in the fundamental representation for $SO(N)$ and $Sp(N)$ on the two sphere. Finally we compare the strong and weak coupling limit of these expressions with those found elsewhere in the literature.

* This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under cooperative agreement #DE-AC02-76ER03069 and by the Division of Applied Mathematics of the U.S. Department of Energy under contract #DE-FG02-88ER25065.

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[†] This work supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under grant #DE-FG02-92ER40706.

SECTION 1. Introduction

Recent work by many authors strongly suggests that pure Yang-Mills gauge theory in two-dimensions is equivalent to a string theory¹⁻¹². The interest in understanding such a simple theory in such great detail is to shed some light on the strong coupling limit of pure QCD in four dimensions. Of course, the string picture of two-dimensional Yang-Mills is interesting in it's own right as a testing ground for non-perturbative analyses of a quantum field theory.

Studying two-dimensional pure Yang-Mills field theory in the large N limit is the starting point for making the correspondence with string theory. For example, as shown in Refs.[1,3,4], a gauge theory based on $SU(N)$ at large N splits into two copies of a “chiral” theory that encapsulates the geometry of the string maps. The chiral theory associated to Yang-Mills theory on a two-manifold \mathcal{M} is a sum over maps from a two-dimensional world sheets (of arbitrary genus) to the manifold \mathcal{M} . This leads to a expansion in $1/N$ for the partition function and observables that converges for all area-coupling product for target space \mathcal{M} of genus one and greater.

The string expansion breaks down for small coupling when the target space is the two-sphere, *i.e.* for Yang-Mills on S^2 . Simply put, for the sphere there are an infinite number of string diagrams that contribute to, for example, the leading term of free energy. Thus, this term, as formulated in terms of string perturbation theory, may fail to converge at a certain value for the coupling. That there must indeed be such a difficulty with the string expansion was first shown by Douglas and Kazakov¹³ (see also Ref.[14].) In particular, they showed that the free energy of QCD₂ on the sphere undergoes a third order phase transition at a value of the coupling $A = A_c = \pi^2$. Extensions of their work appear in Refs.[15-18] in which this relevance of this phase transition was better understood from the point of view of large

N technique and its application to the “chiral” sector of the stringy picture of QCD_2 was carried out. Finally, Boulatov¹⁹, and Daul, et al.²⁰ have computed the expectation value of the Wilson loop (in the fundamental representation) on the sphere using the large N saddle point found in the work of Refs.[13, 14].

In this note we present the large N analysis for QCD_2 on the real projective plane and compute the partition function on the sphere (and on the projective plane) for the classical Lie groups $SO(N)$ and $Sp(N)$ (by convention $Sp(N)$ has rank $N/2$). We also modify the techniques of Refs.[19, 20] to study the expectation value of the Wilson line in the fundamental representation for these groups. We find that at large N it is possible to adapt “technology” developed in Refs.[13-17] to these other cases. We were surprised by the simplicity of the application of these techniques and find an underlying universality that has implications for the observables and for the string interpretation of the theory²².

In section 2 we describe the large N analysis of $U(N)$ QCD_2 on the real projective plane. These results serve as a lemma which allows one to compute the large N phases of the gauge theories $Sp(N)$ and $SO(N)$. These conclusions are presented in section 3. In section 4 we compute the Wilson loop in the fundamental representation on the two sphere for these gauge groups. There we make contact with the known results in strong and weak coupling and further describe the universal analytic form of the Wilson loop and partition function.

SECTION 2: The Phases of $U(N)$ QCD_2 on the Projective Plane

As was described in Refs.[13, 14], the partition sum for $U(N)$ QCD_2 on the sphere is, for large N , dominated by a single representation. In general the partition sum for QCD_2 is

$$Z_{\mathcal{M}} = \int [DA] \exp\left(-\frac{1}{4g^2} \int_{\mathcal{M}} \text{Tr}(F \wedge^* F)\right). \quad (2.1)$$

and specializing to the case where $\mathcal{M} = S^2$, (2.1) has been shown to be*

$$Z_{S^2} = \sum_R \dim(R)^2 e^{-AC_2(R)/2N} \quad (2.2)$$

where $C_2(R)$ is the quadratic Casimir for the representation R and $\dim(R)$ its dimension. In the large N limit it is always possible to write (2.2) in terms of a (quantum mechanical) path integral in a continuous variable h that is linearly related to the row lengths of the Young diagrams for representations

$$Z_{S^2} = \int [Dh] e^{-N^2 S_{eff}(h)}. \quad (2.3)$$

For a fixed gauge group G at large N the partition sum takes the form of (2.3) with h and S_{eff} depending on the group. For reference, note that for the case of $U(N)$ considered previously in Refs.[13,14] one has

$$\dim(R) = \prod_{i < j}^N \left(1 - \frac{n_i - n_j}{i - j}\right); \quad C_2(R) = Nr + \sum_{i=1}^N n_i(n_i - 2i + 1) \quad (2.4)$$

The n_i 's are the row lengths of the Young diagram ($n_i \geq n_{i+1} \ \forall i$) and $r = \sum_i n_i$ is the total number of boxes in the Young diagram for R . For transcription to the large N limit it is convenient to define

$$x = i/N; \quad h(x) = -n(x) + x - 1/2 \quad (2.5)$$

and thus for the case of $U(N)$ one has

$$S_{eff} = - \int_0^1 dx \int_0^1 dy \log(|h(x) - h(y)|) + \frac{A}{2} \int_0^1 dx h(x)^2 + const. \quad (2.6)$$

The action (2.6) has a non-trivial saddle point. This is due to the fact that although the exponential factors in $C_2(R)$ (quadratic in N) quickly become very small, the $\dim(R)$

* In what follows A is the dimensionless area-coupling product $A = g^2 N \text{Area}(\mathcal{M})$. This differs from the definition of \mathcal{A} , the dimensionless area of Ref.[8,10].

factors grow precipitously, the competition between these two factors being responsible for the existence of a non-trivial large N saddle point. Thus, the positive powers of the $\dim(R)$ factors cause the sum to be dominated at large N by a single non-trivial representation R^* . This is special to the two-sphere (and also the projective plane), as the partition sum of QCD_2 on closed surfaces of higher genus will involve non-positive powers of $\dim(R)$.

Furthermore, as shown in Ref.[13], the nature of the saddle point changes dramatically between the small area (A) and large area limits. This change is accompanied by additional terms in the free energy $F = -\frac{1}{N^2} \log(Z)$ that are non-analytic in A . From this point of view, the non-analyticity results from requiring $h(x)$ represent a legal Young diagram (for which $n_i \geq n_{i+1} \ \forall i$). In Ref.[13], for $U(N)$ at large N , that change occurs at an area of $A_c = \pi^2$ and that it is associated with a third order phase transition.

The partition function for QCD_2 for closed non-orientable surfaces is also known^{6,8}. For example, the partition function on the projective plane (\mathbf{RP}^2) is (compare with (2.2))

$$Z_{\mathbf{RP}^2} = \sum_{R=\bar{R}} \dim(R) e^{-\frac{AC_2(R)}{2N}} \quad (2.7)$$

Note that the sum in this case is only over self-conjugate representations. Again, in this sum there will be competition between the dimension factor and the exponential suppression in the Casimirs, and so we expect that there will also be a non-trivial saddle point (i.e. a dominant representation).

At large N this sum again has a simple form in terms of a path integral. Following the example of discussed before, the partition function of $U(N)$ Yang-Mills on \mathbf{RP}^2 may be written as a path integral in the form of (2.3)

$$Z_{\mathbf{RP}^2} = \int' [Dh] \exp \left[-N^2 \left(-\frac{1}{2} \int_0^1 dx \int_0^1 dy \log(|h(x) - h(y)|) \right. \right. \\ \left. \left. + \frac{A}{2} \int_0^1 dx h^2(x) + \text{const}' \right) \right] \quad (2.8)$$

where \int' means that the path integration must necessarily include only those h that represent self-conjugate representations. In (2.9) *ff.* we impose the requirement that the path integral (2.8) involve a sum only over self-conjugate representations. Here we describe another, simpler way of understanding the result we will find below. Note the similarity between the actions of (2.6) and (2.8). Scale out a factor of 2 from the exponent in (2.8) and replace A there by $A/2$. The resulting action is precisely half that of (2.6). Of course, an overall factor will not affect the nature of the saddle point. However, the fact that the sum is only over self-conjugate representations could affect this conclusion profoundly, if it were the case that the dominant representation R^* was not self-conjugate. It is simple to see that this cannot be the case. The contribution from a representation and its conjugate to the sum must be the same: therefore, if there is to be a unique saddle point it must be a self-conjugate representation. Alternatively this may be thought of as a consequence of the \mathbf{Z}_N charge symmetry of the partition function. The large N self-conjugate saddle point R^* remains the dominating saddle point when computing observables (as long as the representations into which the observables decompose have bounded \mathbf{Z}_N charges.) By this reasoning we expect that the phases of the theory on the projective plane will be the same as the phases on the two-sphere, with the only difference being that the phase transition on \mathbf{RP}^2 between the two phases occurs at $A = A_c = \pi^2/2$.

We now directly implement the self-conjugacy condition in the sum Eq.(2.7). Although

as described in the preceding paragraph, we already know the answer, but the direct method we describe is useful for studying QCD₂ based on the other classical Lie groups. The requirement that we sum only over self conjugate representations in $U(N)$ means that there is the additional constraint $n_i = -n_{N-i+1}$ on the rows on the Young diagrams that contribute to the sum. In large N continuum variables this implies that h (see (2.5)) must satisfy

$$h(x) = -h(1-x). \quad (2.9)$$

A simple way to implement this condition is to define a new function g such that

$$h(x) = \begin{cases} g(x) & 0 \leq x \leq 1/2, \\ -g(1-x) & 1/2 \leq x \leq 1, \end{cases} \quad (2.10)$$

the function g being defined only on the interval $[0, 1/2]$. Writing the partition sum (2.7) in terms g allows one to write an ‘unrestricted’ path integral (we apply the measure zero condition $g(1/2) = 0$ later in selecting the solution.)

$$\begin{aligned} Z_{\mathbf{RP}^2} = & \int [Dg] \exp[-N^2(\int_0^{1/2} dx \int_0^{1/2} dy \log(|g^2(x) - g^2(y)|) \\ & + A \int_0^{1/2} dx g^2(x) + \text{const}')] \end{aligned} \quad (2.11)$$

Note the range of the integrations. The saddle point that dominates this path integral is given by the equation of motion

$$P \int ds \frac{u(s)}{g^2 - s^2} = P \int ds' \frac{u'(s')}{g' - s'} = A/2 \quad (2.12)$$

where we have defined $g' = g^2$, $s' = s^2$ and $u'(s')ds' = u(s)ds$. This is one of the very simplest principle-value integral equations to solve²¹ for u . A general solution is

$$u'(g') = \frac{b + \frac{A}{\pi}(a - g')}{\sqrt{g'(2a - g')}} \quad (2.13)$$

for some yet-to-be-determined constants a and b . Here $g' \in [0, 2a]$. Now, using the fact that $\int du(g) = 1/2$, we can determine the constant b

$$u'(g') = \frac{1}{2\pi} \left[\frac{1 + 2A(a - g')}{\sqrt{g'(2a - g')}} \right] \quad (2.14)$$

The solution u must be bounded and this implies that $a = 1/2A$. Now transforming u' back to u we have

$$u(g) = \frac{A}{\pi} \sqrt{\frac{2}{A} - g^2} \quad (2.15)$$

where $g \in [-\sqrt{\frac{2}{A}}, \sqrt{\frac{2}{A}}]$. This is the same solution found for the case of $U(N)$ in Ref.[13], up to a scaling of A by a factor of 2.

Since $h(x)$ must represent a legal Young diagram, we must further require that $u \leq 1$ everywhere. For large areas ($A > \pi^2/2$) the solution (2.15) will have regions for which $u > 1$. Thus it is necessary that we adopt an alternative Ansatz for solving the integral equation (2.12) in that case. Just as in the $U(N)$ case, the double cut Ansatz is applicable, and analysis reveals that again the $U(N)$ form of $u(h)$ is precisely that found in the large A limit of the analysis of Douglas and Kazakov, except for the overall factor of two in the area.

Furthermore, it is easy to check directly that the free energy $F = \log(Z_{\mathbf{RP}^2})$ is simply that found in Ref.[13] but scaled by an overall factor of $1/2$ and $A \rightarrow 2A$. Thus, on \mathbf{RP}^2 at $A = A_c = \pi^2/2$ large N QCD₂ (with gauge group $U(N)$) has a third order phase transition.

SECTION 3: The Large N Analysis of $Sp(N)$ and $SO(N)$

It is possible to take the large N limit of gauge theory for any classical Lie group using the methods discussed in the previous section. At first sight it seems as though $Sp(N)$ and $SO(N)$ gauge theory in the large N limit in two dimensions would be quite different than the case

of $U(N)$ above. The quadratic Casimir and the dimensions of representations differ quite significantly between the groups $U(N)$, $Sp(N)$ and $SO(N)$, and the differences persist in the large N limit. Thus, somewhat surprisingly, we will show in this section that their large N phase structure (up to some simple factors) is identical. The differences between the various groups will first be manifest at non-leading order in $\frac{1}{N}$.

This surprising similarity between the different gauge models suggests a very intriguing large N universality in the partition function and observables of these theories. We discuss this universality in the context of the Wilson loop observable in QCD₂ in section 4, and here study its antecedent in the partition function. Recently, in collaboration with S. G. Naculich, we have been able to understand this universality using the string picture²². From the group-theoretic point of view, universality must thus be a consequence of the "unitarity" (boundedness of the eigenvalues), and not dependent on the orthogonal or symplectic properties of the group elements.

To begin with, recall that the Casimirs for the classical Lie groups are

$$C_2(R) = fN \left[r - U(r) + \frac{T(R)}{N} \right] \quad (3.1)$$

where

$$\begin{aligned} f &= \begin{cases} 1 & \text{for } SU(N) \text{ and } SO(N), \\ \frac{1}{2} & \text{for } Sp(N), \end{cases} \\ U(r) &= \begin{cases} r^2/N^2 & \text{for } SU(N), \\ r/N & \text{for } SO(N), \\ -r/N & \text{for } Sp(N), \end{cases} \\ T(R) &= \sum_{i=1}^{\text{rank} G} n_i(n_i + 1 - 2i) = \sum_{i=1}^{k_1} n_i^2 - \sum_{j=1}^{n_1} k_j^2, \end{aligned} \quad (3.2)$$

and where n_i are the row lengths of the Young diagram (k_i are the column lengths) associated to the representation R . Thus we have $n_i \geq n_{i+1} \geq 0$ for all $1 < i < \text{rank}(G)$. Note that the

solutions found by Douglas and Kazakov in Ref.[13] satisfy a relaxed condition appropriate to $U(N)$, namely that the n_i may be negative. However, by simply shifting the independent variable one finds solutions for $SU(N)$ in which the $n_i \geq 0$. For the cases of $Sp(N)$ and $SO(N)$ we will verify that our solutions do indeed satisfy the constraint $n_i \geq 0$.

In terms of row lengths ²³

$$\dim(R) = \prod_{i < j}^n \frac{(l_i^2 - l_j^2)}{(m_i^2 - m_j^2)} \quad \text{for } SO(2n) \quad (3.3)$$

where $l_i = n_i + n - i$ and $m_i = n - i$ while

$$\dim(R) = \prod_{i < j}^n \frac{(l_i^2 - l_j^2)}{(m_i^2 - m_j^2)} \prod_{i=1}^n \frac{l_i}{m_i} \quad \text{for } Sp(N) \text{ and } SO(2n+1) \quad (3.4)$$

with $l_i = n_i + n - i + 1$ and $m_i = n - i + 1$ for $Sp(N)$ where $N = 2n$ while $l_i = n_i + n - i + 1/2$, $m_i = n - i + 1/2$ for $SO(2n+1)$. Passing to the large N limit, we find a universal form for the S_{eff} for the theories $Sp(N)$ and $SO(N)$

$$S_{eff} = - \int_0^{1/2} dx \int_0^{1/2} dy \log(|l^2(x) - l^2(y)|) + \frac{Af}{2} \int_0^{1/2} dx l^2(x) + const' \quad (3.5)$$

where $l(x)$ is the continuum (large N) limit of l_i/N . This form is analogous to the equation (2.11) for the large N effective action of $U(N)$ -QCD₂ on the real projective plane \mathbf{RP}^2 .

Of course, a crucial difference between the case of $U(N)$ and that of the groups $Sp(N)$ and $SO(N)$ is that, for the later groups, a solution related to an allowed Young diagram must satisfy $n_i > 0$ for all i . However as we previously emphasized on general grounds the solutions discussed in Ref.[13] represent the $Q = 0$ sector of the $U(N)$ representation theory (see Ref.[15] for more details), and furthermore must represent self-conjugate representations. Thus, the dominant representation R^* for all areas A in the $U(N)$ theory on S^2 or \mathbf{RP}^2

satisfies $h(x) = -h(1-x)$, and so $h(1/2) = 0$. Equivalently $g(x)$ of (2.10) is positive over its entire range $0 < x < 1/2$. Since $h(x)$ inherits monotonicity from the n_i we find indeed that the dominant $U(N)$ representation has row lengths that are positive between $[0, 1/2]$ and negative beyond that point.

The transcription of (3.5) from the $U(N)$ case indicates that we need only $1/2$ of the range of the full $U(N)$ saddle point distribution h , which provides the solution appropriate to the groups $Sp(N)$ and $SO(N)$ which have the constraint $n_i > 0$.

It is also possible to compare the expressions for the free energy in these various theories. In the light of the above considerations it is perhaps not surprising that the functional dependence of the free energy on the area in these various theories are equal up to simple factors. Indeed, using the equation of motion above, we find that the derivative of the free-energy is

$$\frac{dF}{dA} = \frac{1}{2} \int dg \, u(g) g^2. \quad (3.6)$$

Thus, we may immediately borrow the results of the analysis of Ref.[13] to ascertain that these other gauge theories on the sphere (or on the projective plane) also have a strong and a weak coupling phase separated by a third order phase transition. A summary of the location of the critical points in the various theories on S^2 is

$$A_c = \pi^2/f \quad (3.7)$$

where f is as defined in (3.1). For \mathbf{RP}^2 , all the critical areas are divided by two.

SECTION 4: Wilson Loops on S^2 at Large N

In the preceding sections we have shown that there is a striking large N universality of the partition functions, phase portraits, and free energies on the two-sphere between the classical

Lie groups. In an effort to extend this observation we present in this section universal formulae for simple Wilson loop observables in QCD₂. Wilson loops for all N in QCD₂ on the sphere were first computed in Ref.[24]. We follow closely the notation, conventions and arguments of Boulatov¹⁹ and Daul, et. al.²⁰ and refer the reader to relevant extensions of these works in Ref.[25].

Consider a Wilson loop in the fundamental representation F dividing the two-sphere of (dimensionless) area A into regions of area A_1 and A_2 respectively. Denote the expectation value of the normalized Wilson line by W_F . In two-dimensions the gauge-theoretic expression for this Wilson loop vacuum expectation value (VEV) is

$$W_F = \frac{1}{\dim(F)Z_{S^2}} \sum_R \sum_{R' \in R \otimes F} \dim(R)\dim(R') e^{-[\frac{A_1}{2N}C_2(R) + \frac{A_2}{2N}C_2(R')]} \quad (4.1)$$

where the second sum is over all the representations $R' \in R \otimes F$. and Z_{S^2} is the partition function (2.1) for the theory on a sphere of area $A = A_1 + A_2$. It is now convenient to separate the parts of the sum that involve terms proportional to $e^{\pm N^2}$ raised to some ($\mathcal{O}(1)$) power from those factors that do not have this N dependence as follows

$$W_F = \frac{1}{\dim(F)Z_{S^2}} \sum_R \dim(R)^2 e^{-\frac{A}{2N}C_2(R)} \sum_{R' \in R \otimes F} \frac{\dim(R')}{\dim(R)} e^{-\frac{A_2}{2N}[C_2(R') - C_2(R)]}. \quad (4.2)$$

The fundamental representation F for the classical Lie groups is denoted by a Young diagram with a single box. Thus, the Young diagram of R' and R differ only by a single box, and so neither the ratio $\frac{\dim(R')}{\dim(R)}$ nor the exponent $(C_2(R') - C_2(R))/N$ lead to terms proportional to some power of $e^{\pm N^2}$. The sum (4.2) over R is again dominated by the same single ‘large N ’ representation, R^* , which is the saddle point for the partition function sum.

Refs.[19,20,24] give an account of the computation of W_F for the group $U(N)$ at large N . We now show that, surprisingly, the large N VEV of the Wilson loop W_F has a universal

form independent of the group. This is strongly reminiscent of the results found in section 3 for partition function. It is a fully non-perturbative result.

Let us briefly review the results of Refs.[19,20]. Specializing to the case of $U(N)$ at large N , and adopting the notation $\phi_i = i - n_i - N/2$, it is clear that W_F is

$$W_F = \frac{1}{N} < \sum_j \exp \left[\sum_i \log \left(1 - \frac{1}{\phi_j - \phi_i} \right) + \phi_i A_2 \right] > \quad (4.3)$$

where the sum on the index j represents all the rows of the Young diagram of R^* that one can add a single box to obtain an allowed diagram*. As described in the literature, the sum on i in (4.3) is best approximated at large N by sums over two regions; those in which the box is added close to the j 'th row $|j - i| \leq \sqrt{N}$, and the other in which the single box is put comparatively far from the j 'th row, $|j - i| \geq \sqrt{N}$. The first region contributes an overall factor of $\frac{\sin(\pi u)}{\pi u}$, and the second, by virtue of the equation of motion for R^* , combines with the exponential term that is already present, to yield,

$$W_F = \frac{1}{N} \sum_j \exp \left[\frac{l_j \delta A}{2} \right] \frac{\sin(\pi u)}{\pi u} \quad (4.4)$$

where $\delta A = A_1 - A_2$, and the sum is again over only those rows j of the Young diagram for R^* that can legally admit another box. Actually computing the sum over j requires a little care. This is due to the presence of rows to which one cannot add even a single box. Taking account for such cases is well described in Refs.[19,20] and it finally simply changes an overall factor in the sum. For $Sp(N)$ and $SO(N)$ we will have to address the very same problem. In these cases it is similarly simple to prove that accounting for these rows again simply results in a change in the overall factor of the expression in the sum.

* Equivalently, one could average over R' , with one box *removed* from R^* , and A_1 and A_2 interchanged, with the same result^{19,20}. Note physically the VEV of W_F on the sphere must be symmetric under $A_1 \leftrightarrow A_2$. These observations will be relevant to our discussion of $Sp(N)$ and $SO(N)$.

We now compute the VEV of a Wilson loop (in the fundamental representation F) for the groups $Sp(N)$ and $SO(N)$. Starting again with (4.2) and specializing to the case of $Sp(N)$ ($\eta = 1$, $f = 1/2$; for $SO(N)$ take $\eta = -1$ and $f = 1$) we find

$$W_F(Sp(N)) = \frac{1}{N} < \sum_{R'} \frac{\dim(R')}{\dim(R^*)} \exp \left[-\frac{A_1 f}{2N} (C_2(R') - C_2(R^*)) \right] > \quad (4.5)$$

In (4.5), $<>$ again denotes computing this average on the dominant representation, R^* , of the $Sp(N)$ large N partition function on the sphere, while $\sum_{R'}$ is over all representations that result by fusing R^* and the fundamental representation F . For $Sp(N)$ this means the sum is over all rows of the Young diagram of R^* for which one can add one box *and* for which one can subtract one box, so as to obtain a legal Young diagram. The reason for this is that $Sp(N)$ has an anti-symmetric invariant which allows (anti-symmetric) contractions in the tensor product of the fundamental representation with any representation (a simple example in $Sp(N)$: $\square \otimes \square = \square\square + \square\square + I.$) As a consequence of using (3.3) and (3.4) for the dimensions of representations, one has in the large N limit

$$W_F = \frac{1}{N} < \sum_j \prod_{i \neq j} \left(1 + \frac{1}{l_i - l_j}\right) \left(1 + \frac{1}{l_i + l_j}\right) \exp\left(-\frac{A_2 f}{2N} l_j\right) > + \{A_2 \leftrightarrow A_1\} \quad (4.6)$$

where the first term in (4.6) comes from adding one box and the second term from removing one box. Both terms in (4.6) do indeed give the same function (see the previous footnote). However, as discussed in section 3, the constraint $n_i \geq 0$ for $Sp(N)$ means that we have only $1/2$ of the range of the full $U(N)$ saddle point distribution $h(x)$. Because of the symmetry of the saddle point of $U(N)$, taking $1/2$ the range results in an overall factor of $1/2$ that just cancels the overall factor of 2 in (4.6), and so the Wilson loop for $Sp(N)$ is identical in form to that of (4.4) with $A \rightarrow fA$. Also, note that the question of nearby and distant regions in the sum may be treated in exactly the same way as that of $U(N)$.

The analysis of Wilson loops for $SO(N)$ is identical. For $SO(N)$ fusions one recalls that there is a symmetric (bilinear) invariant; one must therefore again sum over all rows of R^* to which one can legally add one box *and* subtract one box. As a result the Wilson loop in the fundamental representation (for all the classical Lie groups) at large N on S^2 can be written in a universal way as

$$W_F = \frac{1}{N} \sum_j \frac{\sin(\pi u)}{\pi u} \exp \left[\frac{l_j f \delta A}{2} \right] \quad (4.7)$$

where l_j is defined for $U(N)$ above (4.3) and in (3.3), (3.4) for $SO(N)$, $Sp(N)$ respectively (see also (3.2)) replacing the sum by a continuous integral results in the final universal result

$$W_F(G; A_1, A_2) = \int d\phi \frac{\sin(\pi u)}{\pi} \exp \left[\frac{f}{2} (A_1 - A_2) \phi \right] \quad (4.8)$$

where $\phi(u)$ is the continuous (large N) analogue of l_j . Note that in (4.8) the integration is over the whole range of $\phi(u)$, due to the two identical terms in (4.6) for $Sp(N)$ (as for $SO(N)$).

This proves the rather surprising result that there is a universal large N expression for the expectation value of a Wilson line (in the fundamental representation on the two-sphere) that is independent of the gauge group. This is expected for the small coupling phase for the perturbative Wilson loop VEV but, we emphasize, here we have shown a much stronger statement, namely that the equality is a non-perturbative result true for all coupling. In the large area phase, W_F is expressible as an expansion in polynomials of $[A_i, e^{-A_i/2}]$ $i = 1, 2$. Now, in the light of how different the dimensions and the Casimirs are between the various classical Lie groups, we might expect that the actual expansions in $[A_i, e^{-A_i/2}]$ $i = 1, 2$ will differ between these groups. However, because of the preceding analysis we know that there

is indeed a universal formula (4.8) which evaluated in the large A_i limit reads *

$$\begin{aligned}
W_F = & e^{-A_2/2} + (-1 - A_2 + A_2^2/2)e^{-(2A_2+A_1)/2} \\
& + (1 + A - \frac{A_1^2}{2} + \frac{3A_2^2}{2} + A_1A_2 - \frac{8A_2^3}{3} - \frac{A_1^2A_2}{2} - \frac{5A_1A_2^2}{2} + \frac{2A_1A_2^3}{3} + \frac{A_1^2A_2^2}{4} + \frac{A_2^4}{2})e^{-(3A_2+2A_1)/2} \\
& + \dots + (A_2 \leftrightarrow A_1)
\end{aligned} \tag{4.9}$$

for both $SU(N)$ and $SO(N)$ (also for $Sp(N)$ after the trivial scaling $A_i \rightarrow fA_i$.) The fact that these Wilson loop VEV's are the same (regardless of the group) in the strong coupling regime is not at all obvious from group theoretic considerations, or the perturbative expansion of the theory (since, at large N , we know we cannot follow the usual perturbation expansion into the strong coupling regime.) It is simple to expand out the gauge theory result about large A_i to check (4.9). The result is identical to the gauge-theoretic normalized Wilson loop VEV formula at large A_i (see for example section 5 of Ref.[10]). From the gauge theoretic expression, universality at large A_i results from an intricate conspiracy between the $\dim(R)$ factors and the quadratic Casimirs. It is *not* simply a fact about the leading large N part of the $\dim(R) \sim d_R N^r / r! + \dots$ (in notation of Ref.[8]) but is a result that involves all the sub-leading terms in the dimension formulae. Of course, it would be most satisfying to have a deeper understanding of this universality. With S. Naculich, we have recently completed proof of this equivalence, describing it in the string picture as an isomorphism between string maps in the various theories. This proof of large N universality will be presented elsewhere²².

SECTION 5: Conclusion

We have shown the partition function and certain observables on the two-sphere in QCD_2 at large N are essentially independent of the underlying gauge group. This is not a trivial

* To compare with the strong coupling result of Boulatov¹⁹ see appendix.

consequence of the large N limit, but emerges as consequence of detailed analysis, and is true for all values of the coupling-area product. Such an equivalence is not obvious from the string point of view but implies an equivalence between classes of maps for any classical Lie group.

As described in forthcoming work²², the string picture of two-dimensional Yang-Mills gauge theories does provide the most satisfying understanding of the universality discussed in this present paper. This should be considered a major success of the string picture of QCD_2 as it gives insights not readily available otherwise.

Acknowledgements

We thank S. G. Naculich, H. A. Riggs and W. Taylor for comments.

Appendix

This appendix is devoted to amending a few of the formulae in Ref.[19]. They regard the computation of the Wilson loop VEV at strong coupling.

In the strong coupling phase, much of the analysis concerning the large N VEV of a Wilson loop in the fundamental representation involves expansions of elliptic functions. A reference on elliptic functions that takes a very practical approach (ideal for one checking the formulae in Ref.[19]) is Lawden's book²⁶. We will be brief in this appendix but refer the interested reader to that reference for some of the details.

In Ref.[19] the second line of equation (50) is in error. Let $A = A_1 + A_2$. We find that in the large coupling limit ($A \rightarrow \infty$ is the $\tilde{q} = e^{-\frac{\pi a A}{4K'}} \rightarrow 0$ limit, in the variables of Ref.[19])

$$\frac{2K'}{\pi} \frac{dn(2K'v, k')}{sn^2(2K'v, k')} = \frac{1}{\sin^2(\pi v)} [1 - 4\tilde{q} + 8\tilde{q}^2 + 4\tilde{q}^2(\cos(2\pi v) + \sin^2(2\pi v) + \dots)] \quad (A.1)$$

All the other formulae in (50) of Boulatov¹⁹ are correct, but note the factor of π difference between the argument of the Θ -functions in Ref.[19] versus Ref.[26].

With this correction we modify (51) of Ref.[19] for the VEV of the Wilson loop on the sphere to read

$$\begin{aligned} W_F = -i\pi a \oint \frac{dv}{2\pi i} \frac{\exp\left(\frac{2\pi i K A_2}{K' A} \cot(\pi v) + 2\pi i v\right)}{\sin^2(\pi v)} & \left[1 - 4\tilde{q} + \frac{8\pi i K A_1}{K' A} \tilde{q}^2 \sin(2\pi v) \right. \\ & \left. + 8\tilde{q}^2 + 4\tilde{q}^2 (\cos(2\pi v) + \sin^2(2\pi v)) + \dots\right] \end{aligned} \quad (A.2)$$

Note that the saddle point implies that $4K = Aa$ and now define $\beta = 1 + 8\tilde{q}^2 \log \tilde{q}$. We then have

$$\frac{a}{K'} = \frac{1}{\pi} (1 + 8\tilde{q}^2 \log \tilde{q} + \dots) = \frac{1}{\pi} \beta + \dots \quad (A.3)$$

Now, under the variable change $x = i \cot(\pi v)$ and also inverting the relation for a from equation (50) Boulatov

$$a = \frac{1}{2} (1 + 4\tilde{q} + 4\tilde{q}^2 + 8\tilde{q}^2 \log \tilde{q} + \dots), \quad (A.4)$$

we may rewrite the Wilson loop VEV as

$$\begin{aligned} W_F = \frac{1}{2} \oint \frac{dx}{2\pi i} \left(\frac{1-x}{1+x} \right) e^{A_2 \beta x/2} & [1 - 4\tilde{q}^2 + 4A_1 \tilde{q}^2 \left(\frac{x}{1-x^2} \right) \\ & + 8\tilde{q}^2 \log \tilde{q} - 4\tilde{q}^2 \left(\frac{1+x^2}{1-x^2} + \frac{4x^2}{(1-x^2)^2} \right) + \dots] \end{aligned} \quad (A.5)$$

For comparison see equation (51) of Ref.[19]. To the order that we are working then we may replace all occurrences of $\log \tilde{q}$ by $-A/4$. We must also be careful about keeping the $\mathcal{O}(\tilde{q}^2)$ terms from the exponent $e^{-A_2 \beta/2} = e^{-A_2/2} (1 + AA_2 \tilde{q}^2)$ to the order that we are working. Putting all of this together and computing the Wilson loop VEV by summing the residues of the integral (A.5), we find to this order the result (4.9), as expected.

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